Minimum Genus Embeddings of Vertex-Transitive Graphs

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Quick introduction to graph embeddings

A 2-cell embedding of a graph Γ on a surface S is a drawing of Γ on S without edge-crossings (i.e. such that arcs representing the edges of Γ have no crossings except when two edges with a common vertex meet at that vertex), and such that removal of the graph breaks up the surface into simply-connected regions, called the faces of the embedding.

The characteristic of any such embedding is the Euler characteristic χ of the carrier surface S, which is given by the Euler-Poincaré formula $\chi = V - E + F$, where V, E and F are the numbers of vertices, edges and faces.

Any graph embeddable on the sphere is called planar.

The genus of a graph embedding

The genus of any graph embedding is the genus g of the carrier surface, which is given by

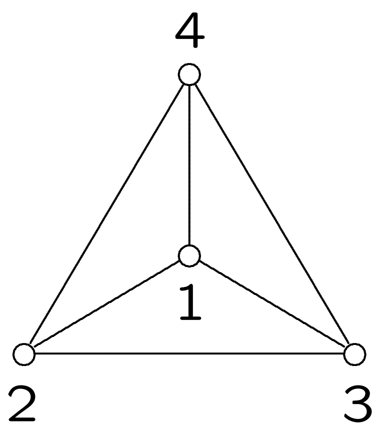
$$\chi = \begin{cases} 2 - 2g & \text{if the surface is orientable} \\ 2 - g & \text{if the surface is non-orientable.} \end{cases}$$

(Essentially, the genus is the number of handles or cross-caps added to the sphere to obtain the surface.)

The minimum genus $\gamma(\Gamma)$ of a graph Γ is the smallest genus of all of its embeddings on orientable surfaces (occurring when the number of faces is as large as possible), and similarly, the maximum genus $\gamma_M(\Gamma)$ is the largest such genus (occurring when the number of faces is as small as possible).

Rotation systems

Every embedding of a graph Γ on an orientable surface S is uniquely determined by the cyclic orientation of the edges at each vertex. The rotation system can be used to determine the faces of the embedding, and hence the genus, e.g.:



Rotations

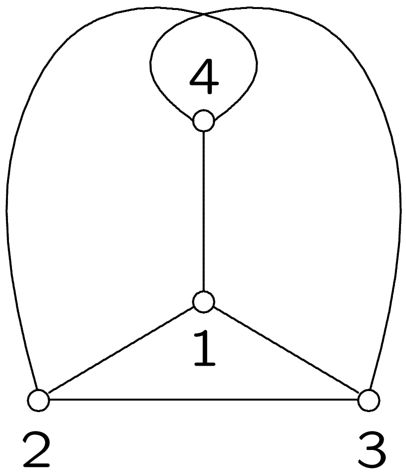
1: $2, 3, 4$
2: $1, 4, 3$
3: $1, 2, 4$
4: $1, 3, 2$

Faces $[1, 2, 3]$, $[1, 3, 4]$, $[1, 4, 2]$, $[2, 4, 3]$

Characteristic $\chi = 4 - 6 + 4 = 2$

Genus 0 (planar)

Changing a rotation alters the embedding:



Rotations

1: $2, 3, 4$
2: $1, 4, 3$
3: $1, 2, 4$
4: $1, 2, 3$

Faces $[1, 2, 3]$, $[1, 3, 4, 2, 1, 4, 3, 2, 4]$

Characteristic $\chi = 4 - 6 + 2 = 0$

Genus 1 (embedding on the torus)

Maximum genus of a graph

It was proved by Xuong (1979) that the maximum genus of an arbitrary finite graph Γ is given by

$$\gamma_M(\Gamma) = \frac{1}{2}(\beta(\Gamma) - \xi(\Gamma)),$$

where $\beta(\Gamma)$ is the Betti number of Γ, which is $1 - V + E$, and $\xi(\Gamma)$ is the deficiency of Γ.

The latter quantity is defined as follows. The deficiency $\xi(\Gamma, T)$ of a spanning tree T of the graph Γ is the number of components of $\Gamma \setminus T$ that have an odd number of edges, and then the deficiency $\xi(\Gamma)$ of Γ is the minimum value of $\xi(\Gamma, T)$ over all spanning trees of Γ.

Maximum genus of VT graphs

Martin Škoviera and Roman Nedela (1989) used theorems by Xuong et al (on embeddings of 4-edge-connected graphs) to prove that a finite connected vertex-transitive graph is upper-embeddable whenever its valency or girth is at least 4.

The only such VT graphs that are not upper-embeddable are 3-valent examples of girth 3 (and order 18 or more).

It then follows that every connected finite Cayley graph is upper-embeddable.

Next: What about **minimum genus?**

Minimum genus of VT graphs

In contrast to the maximum genus, relatively little is known about the minimum genus of embeddings of vertex-transitive graphs (in orientable and non-orientable surfaces).

There are some famous infinite families of VT graphs for which the minimum orientable genus is known, e.g.:

- Cycle graph C_n and other Platonic graphs ... genus 0
- Complete graph K_n ... genus $\lceil\frac{(n-3)(n-4)}{12}\rceil$
  (Ringel & Youngs, 1968)
- Complete bipartite graph $K_{n,n}$ ... genus $\lceil\frac{(n-2)^2}{4}\rceil$
  (Ringel, 1965)
- Hypercube Q_n ... genus $(n-4)2^{n-3}+1$
  (Beineke & Harary, 1965).

Another infinite family

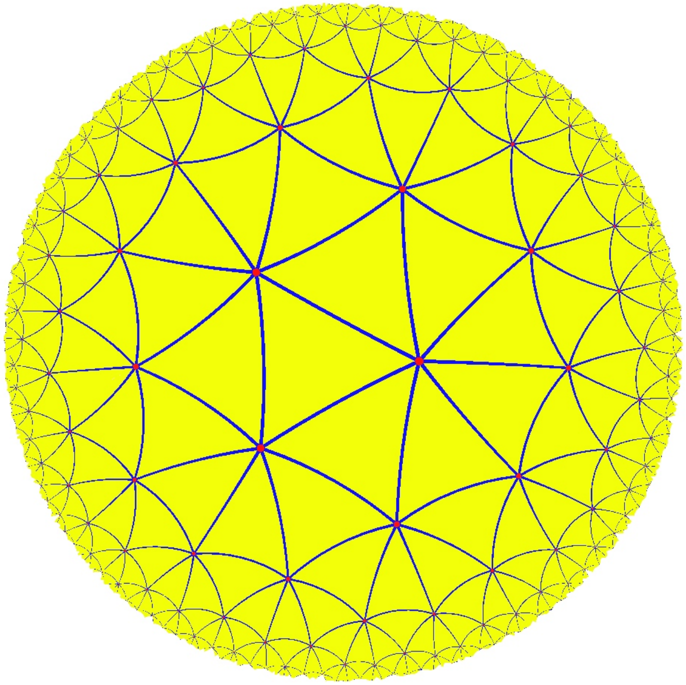
Let G be a (finite) Hurwitz group – generated by elements x and y such that x, y and xy have orders 2, 3 and 7.

Then G is the orientation-preserving group of automorphisms of a regular map of type $\{7,3\}$ on an orientable surface of genus $g > 1$, where $|G| = 84(g-1)$, the maximum possible number of conformal automorphisms of such a surface.

The Cayley graph on generating set $\{x, y, y^{-1}\}$ for G has an embedding on the same surface, and contracts to the map by replacing the 3-cycles for the relator y^3 by single vertices. These give minimum genus embeddings of both the Cayley graph and the underlying graph of the map, as there are no shorter relators than the obvious ones (x^2, y^3 and $(xy)^7$).

There are infinitely many examples of such 'Hurwitz' maps, each giving a quotient of the universal $\{7,3\}$ tessellation of the hyperbolic plane

... dual to the $\{3,7\}$ tessellation below:



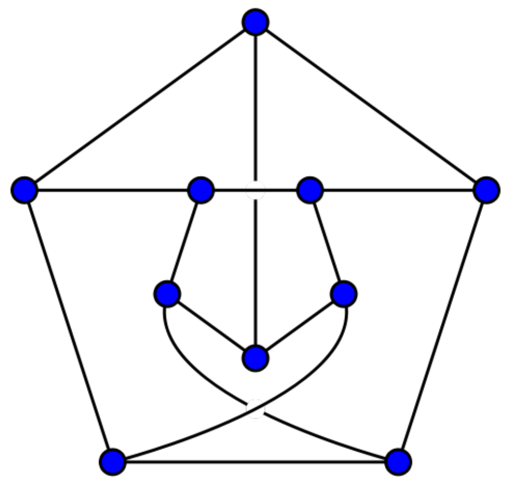
Minimum genus of other VT graphs

Jungerman & White (1980) determined the minimum genus $\gamma(A)$ of all orientable embeddings of all Cayley graphs for a given finite abelian group A, in a large number of cases.

The minimum orientable genus has also been found for a small number of sporadic examples of VT graphs, e.g.:

- Petersen graph ... genus 1
- Heawood graph ... genus 1
- Möbius-Kantor graph ... genus 1
  (Marušič & Pisanski, 2000)
- $C_3 \times C_3 \times C_3$ ... genus 7 (Brin & Squier, 1988).
- Cayley graphs for some other small groups.

The Petersen graph is non-planar since it has K_5 as a graph minor, but it can be drawn on the plane/sphere with just two edge-crossings:



It can also be drawn on the torus with no edge-crossings and five faces (e.g. faces of lengths 5, 5, 5, 6 and 9), and so the minimum orientable genus of the Petersen graph is 1.

The minimum genus of non-orientable embeddings of the Petersen graph is 1, given by a 2-cell embedding on the projective plane with six pentagonal faces, as a regular map of type $\{5, 3\}$, and automorphism group A_5:

(Take the standard drawing of the Petersen graph and place a cross-cap within the five-point star at the centre.)

Finding the minimum genus of a graph

A lot of attention has been paid to the graph genus problem, which is the problem of deciding whether a given connected graph Γ has an orientable embedding of given genus g.

Filotti, Miller and Reif (1979) described an algorithm for finding such an embedding when one exists, intended to run in $|V(\Gamma)|^{O(g)}$ steps – but its validity has been questioned.

Thomassen (1989) showed that the graph genus problem is NP-complete. More generally, finding the minimum genus of a graph is NP-hard.

Also Thomassen (1991) proved a conjecture of Babai, that for every $g \geq 3$, there are only finitely many vertex-transitive graphs with minimum genus g.

Finding the minimum genus of a graph (cont.)

Mohar (1999) developed a linear-time algorithm that finds for a given graph Γ and surface S, either an embedding of Γ in S, or a subgraph of Γ that is a minimal forbidden subgraph for embeddability in S (i.e. a minor for S).

This yields a constructive proof of the Robertson-Seymour theorem about every surface S having finitely many minors, but from a practical standpoint, the algorithm works well only when S has very small genus (e.g. 0 and 1).

Still, we can sometimes exploit symmetry to find the minimum genus of a VT graph, even when that genus is large.

Circulants

A circulant is a Cayley graph $C_n(X) = \mathrm{Cay}(\mathbb{Z}_n, X)$ for a cyclic group. Examples are as follows:

Simple cycle C_n: $X = \{1\}$, edges $\{a, a+1\}$ for all $a \in \mathbb{Z}_n$

Complete graph K_n: $X = \{1, 2, \ldots, n-1\}$, all possible edges

Complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$: $X = \{1, 3, 5, n-1\}$, even n

Paley graph $P(q)$: $X =$ set of non-zero squares in $\mathrm{GF}(q)$

$K_n - \frac{n}{2}K_2$ (Complete graph minus a perfect matching):
$X = \{1, 2, \ldots, n-1\} \setminus \{\frac{n}{2}\}$, when n is even

Note: If $X = \{a_1, \ldots, a_k\}$, we write $C_n(a_1, \ldots, a_k)$ for $C_n(X)$.

Planar circulants

The question of planarity of connected circulant graphs was settled completely by Heuberger (2003).

If X is 'properly given' (with $y \neq \pm x$ for every two $x, y \in X$) and $C_n(X)$ is connected, then $C_n(X)$ is planar if and only if

(a) $|X| = 1$ (and $C_n(X)$ is a simple cycle), or

(b) n is even, and $X = \{x, y\}$ where $y \equiv \pm 2x$ mod n, or

(c) n is even, and $X = \{x, \frac{n}{2}\}$ where x is even.

In particular, $C_n(X)$ is non-planar whenever $|X| \geq 3$.

Embeddings on the torus (genus 1)

Embeddings of circulant graphs on the torus were investigated by Costa, Strapasson, Alves and Carlos, and reported in a 2010 preprint.

Costa et al. claimed that if X is 'properly given' and $C_n(X)$ is connected, then $C_n(X)$ has minimum genus 1 iff

(a) $|X| = 2$ and $C_n(X)$ is non-planar, or

(b) $|X| = 3$ and $X = \{x, y, x+y\}$, or

(c) $|X| = 3$ and $X = \{x, 2x, \frac{n}{2}\}$ with x and $\frac{n}{2}$ odd, or

(d) $n = 8$ and $X = \{1, 2, 4\}$.

... but they made mistakes in their analysis (see later).

More on minimum genus (of regular graphs)

For minimum genus, we need to maximise F (the number of faces), and therefore minimise the average face-size.

This can often be found by trying to get as many triangular faces as possible.

Counting incident edge-faces pairs in two different ways gives $3F \leq 2E = dn$, where d is the valency, and it follows that for genus 1 or 2 we need

$$-2 \;\leq\; \chi \;=\; V - E + F \;\leq\; n - \frac{dn}{2} + \frac{dn}{3} \;=\; \frac{(6-d)n}{6}$$

and so $(d-6)n \leq 12$, which gives upper bounds of $d \leq 7$ and $|X| \leq 4$ for small n, and $d \leq 6$ and $|X| \leq 3$ for large n.

Circulant embeddings of small genus

[Joint work with Ricardo Grande, 2012]

We took a theoretical approach, considering what triangular faces are possible at a vertex and its neighbours, for a given generating set X. We also did some experimentation by computer, using both systematic and random searches.

This combination of approaches helped us to find a counter-example (and then an infinite family of counter-examples) to the 'theorem' of Costa, Strapasson, Alves and Carlos, and also enabled us to find all connected circulants with minimum genus 1 or 2.

Example A: $n = 12 \quad X = \{1, 3, 4, 6\}$

Can this have a genus 2 embedding? By Euler-Poincaré, we would need $-2 = 2-2g = \chi = V - E + F = 12 - 42 + F$, so $F = 28$, with average face-size $2E/F = 84/28 = 3$, which means a triangulation.

The only triangles with edge $\{0, 1\}$ are $\{0, 1, 4\}$ and $\{0, 1, 9\}$, and the only triangles with edge $\{0, 4\}$ are $\{0, 4, 1\}$, $\{0, 4, 3\}$ and $\{0, 4, 8\}$, etc., so for a triangulation, the only possible rotations at vertex 0 are $(1, 9, 6, 3, 11, 8, 4)$ and its reverse, $(1, 4, 8, 11, 3, 6, 9)$. Similarly, the possible rotations at vertex 3 are $(4, 0, 9, 6, 2, 11, 7)$ and its reverse, $(4, 7, 11, 2, 6, 9, 0)$.

But from the rotation at 0 we find $[0, 3, 6]$ or $[0, 6, 3]$ is a face, but from the rotation at 3 we see this is impossible.

Example B: $n = 12 \quad X = \{1, 2, 5\}$

Can this have a genus 1 embedding? By Euler-Poincaré, we would need $0 = 2 - 2g = \chi = V - E + F = 12 - 36 + F$, so $F = 24$, with average face-size $2E/F = 72/24 = 3$, which again means a triangulation.

The triangles with edge $\{0, 1\}$ are $\{0, 1, 2\}$ and $\{0, 1, 11\}$, and the triangles with edge $\{0, 2\}$ are $\{0, 2, 1\}$ and $\{0, 2, 7\}$, etc., so for a triangulation, the only possible rotations at vertex b are $(b+1, b+2, b+7, b+5, b+10, b+11)$ and its reverse, $(b+1, b+11, b+10, b+5, b+7, b+2)$.

Now taking the first rotation for all even b and the second rotation for all odd b gives such an embedding on the torus.

The same happens whenever $X = \{1, 2, \frac{n}{2} - 1\}$, for $\frac{n}{2}$ even.

Some other interesting/tricky cases

- $C_8(1,2,3,4)$ has minimum genus 2

Why? This is K_8, and in their proof of the Heawood Map Colouring Problem, Ringel and Youngs (1968) showed that the minimum genus of the complete graph K_n is $\lceil \frac{(n-3)(n-4)}{12} \rceil$.

- $C_{10}(1,2,4)$ has minimum genus 2

Why? One subgraph is $C_{10}(2,4)$, which is the union of two copies of K_5, and since K_5 has minimum genus 1, it follows from a theorem of Battle, Harary, Kodama and Youngs (1962) that $C_{10}(1,2,4)$ has genus at least $1+1=2$.

- $C_{20}(1,5,10)$ has minimum genus greater than 2

Why? Small genus requires getting two triangular faces at many vertices, and those triangular faces have to be bounded by triples of the form $\{i, i+5, i+10\}$, but then the edges $\{i, i \pm 1\}$ are forced to lie in a face of length 5 or more, which makes the genus too large.

- $C_{10}(1,2,4,5)$ has minimum genus greater than 2

This one needed a computation to check several possibilities.

- $C_{11}(1,2,4)$ has minimum genus greater than 2

This one needed a lot of computation to check several possibilities in a number of sub-cases.

Circulants with minimum genus 1

Up to isomorphism, the connected circulants with minimum genus 1 are as follows:

- $C_n(a_1, a_2)$ when this is non-planar;
- $C_n(a_1, a_2, a_3)$ when $a_3 = \pm(a_1 + a_2)$;
- $C_n(1, 2, \frac{n}{2})$, when $n \equiv 2 \pmod 4$ and $n \geq 10$;
- $C_n(1, 2, \frac{n}{2} - 1)$, when $n \equiv 0 \pmod 4$ and $n \geq 12$;
- $C_8(1, 2, 4)$ and $C_9(1, 2, 4)$.

Circulants with minimum genus 2

Up to isomorphism, the connected circulants with minimum genus 2 are as follows:

- $C_n(1, 2, \frac{n}{2})$, when $n \equiv 0 \pmod 4$ and $n \geq 12$;
- $C_n(1, 2, \frac{n}{2}-1)$, when $n \equiv 2 \pmod 4$ and $n \geq 10$;
- $C_n(2, 4, \frac{n}{2})$, when $n \equiv 2 \pmod 4$ and $n \geq 10$;
- $C_{12}(1, 2, 4)$, $C_{12}(1, 3, 6)$, $C_{12}(1, 4, 6)$, $C_{12}(2, 3, 6)$, $C_{12}(3, 4, 6)$, $C_8(1, 2, 3, 4)$ and $C_{12}(1, 4, 5, 6)$.

The Hoffman-Singleton graph

The Hoffman-Singleton graph is the Moore graph of valency 7 and diameter 2. As such, it has $1+7+7\cdot 6 = 50$ vertices, and $7\cdot 50/2 = 175$ edges, and girth 5. It is vertex-transitive, and indeed 3-arc-transitive, but not a Cayley graph.

It automorphism group is isomorphic to $\mathrm{P\Sigma U}(3,5)$, of order 252000. The stabiliser of a given vertex v is isomorphic to S_7, and acts faithfully on the neighbourhood of v.

But its automorphism group has no subgroup of order 175 or 350, so the Hoffman-Singleton graph is not the underlying graph of an edge-transitve or regular map.

Theorem [Klara Stokes & MC, proved in 2014]:

The minimum genus of non-orientable embeddings of the Hoffman-Singleton graph is 57, via an embedding with 70 pentagonal faces (and Euler characteristic -55).

In fact, there exist such embeddings with map automorphism group isomorphic to C_5 or C_7, but of no larger order.

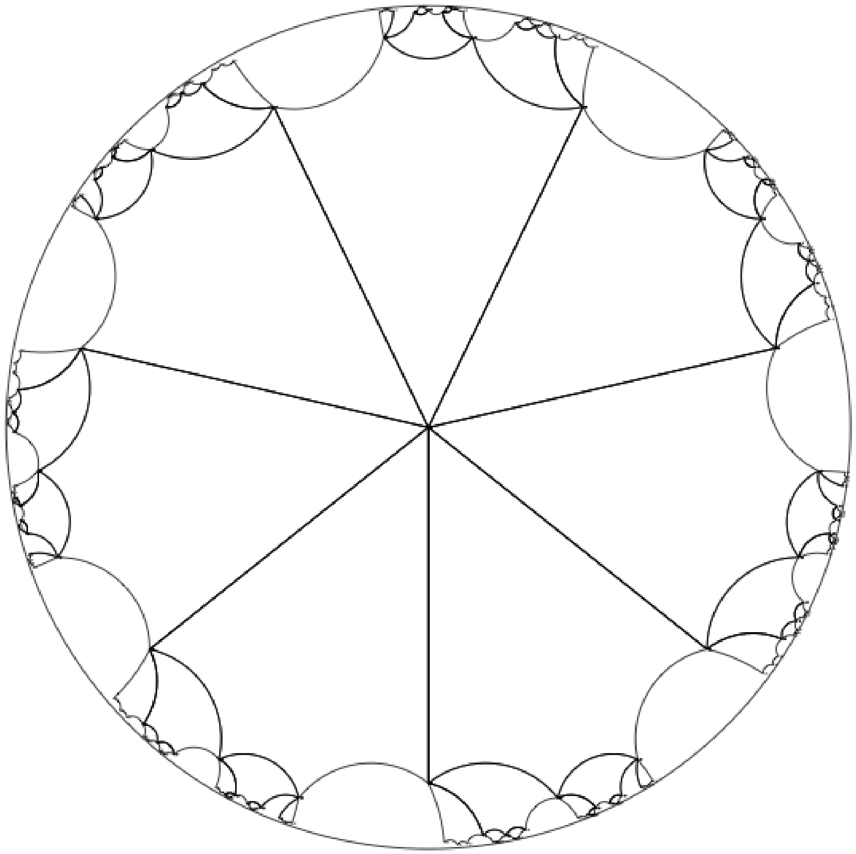
Proof method. Take a graph automorphism θ, and consider orbits of $S = \langle\theta\rangle$ on undirected circuits of length 5. Then systematically determine all ways in which a subset of these can be combined together to give 70 faces of an embedding, with valid 'local' rotation at each vertex. This can be done when θ has order 5 or 7.

Hoffman-Singleton graph (cont.)

We can take θ as one particular automorphism of order 5 that fixes no vertex, and find 14 orbits of $\langle\theta\rangle$ of length 5 that combine together to give the 70 faces of a minimum genus non-orientable embedding, with map automorphism group $S = \langle\theta\rangle$ of order 5.

Or we can take θ as any automorphism of order 7, which fixes just one vertex, and find 10 orbits of $\langle\theta\rangle$ of length 7 that combine together to give the 70 faces of a minimum genus non-orientable embedding, with map automorphism group $S = \langle\theta\rangle$ of order 7. See the picture [KS] following.

We also found one with trivial automorphism group.



A minimum genus non-orientable embedding of the Hoffman-Singleton graph with 7-fold symmetry

Theorem [Klara Stokes & MC, proved in 2014]:

The minimum orientable genus of the Hoffman-Singleton graph is 29, coming from an embedding with 69 faces (and Euler characteristic -56).

In fact, there exist such embeddings with map automorphism group of order 5, but of no larger order.

Proof method. As for the non-orientable case, but requiring a consistent rotation system (across all 50 vertices). One such embedding has 64 pentagonal faces and 5 hexagonal faces, with automorphism group of order 5.

Thank you

Title: Minimum genus embeddings of vertex-transitive graphs
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Abstract:

By a theorem of Škoviera and Nedela (1989), almost all vertex-transitive graphs are 'upper-embeddable', in that they have 2-cell embeddings on orientable surfaces of maximum conceivable genus, with just one or two faces. (The only exceptions are 3-valent examples of girth 3 and order 18 or more.) In particular, every finite connected Cayley graph is upper-embeddable.

In contrast, relatively little is known about the *minimum* genus of vertex-transitive graphs. Finding the minimum

genus of a given connected graph is a notoriously difficult problem, except in some very special circumstances (such as when the graph is planar, or is a Cayley graph for some quotient of the $(2, 3, 7)$ triangle group).

In this talk, I will describe two recent developments on this topic. One is some work with Ricardo Grande in 2012/13, on finding the minimum genus of families of connected circulants (Cayley graphs for cyclic groups), including a complete determination of all such graphs that have minimum genus 0, 1 or 2. The second is joint work in 2014 with Klara Stokes, on exploiting symmetries to find the smallest genus of embedding of the Hoffman-Singleton graph, in both the orientable and non-orientable cases, with 69 faces and 70 pentagonal faces respectively.